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Stechkin–Marchaud-Type Inequalities for Baskakov Polynomials[1]

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E. Van Wickeren (1986, *Constr. Approx.* **2**, 331–337) shows some Stechkin–Marchaud-type inequalities in connection with Bernstein polynomials. In this paper, we introduce $\omega^2_{\varphi^\lambda}(f, t)_{\alpha, \beta}$, and give the Stechkin–Marchaud-type inequalities

type inequalities.

1. INTRODUCTION

For the Bernstein polynomials

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \tag{1.1}$$

in [2] Ditzian gave an interesting direct estimate,

$$|B_n(f, x) - f(x)| \leqslant C\omega^2_{\varphi^\lambda}\left(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right), \qquad 0 \leqslant \lambda \leqslant 1,\ \varphi(x) = \sqrt{x(1-x)}, \tag{1.2}$$

which unifies the classical estimate for $\lambda = 0$ and norm estimate for $\lambda = 1$.

As the inverse results, [7] obtains the Stechkin–Marchaud-type inequalities for Bernstein polynomials as follows

$$\omega^2_\alpha\left(f, \frac{1}{\sqrt{n}}\right) \leqslant M n^{-1} \sum_{k=1}^{n} \|B_k f - f\|_\alpha \qquad (0 \leqslant \alpha \leqslant 2), \tag{1.3}$$

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where $\omega_\alpha^2(f,t)=\sup\{\varphi^{-\alpha}(x)\,|\Delta^2_{h\varphi(x)}f(x)|: x, x\pm h\varphi(x)\in[0,1], 0<h\leqslant t\}$, $\varphi(x)=\sqrt{x(1-x)}$, $\Delta^2_{h\varphi(x)}f(x)=f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x))$ and $\|f\|_\alpha:=\|\varphi^{-\alpha}f\|_{C[0,1]}$. But, this is only a norm estimate (with $\omega^2_\varphi(f,t)$), the classical estimate (with $\omega^2(f,t)$) is not included.

In [3] Ditzian and Ivanov gave the strong converse inequality: for the Bernstein operator there is a k such that

$$\omega^2_\varphi\left(f,\frac{1}{\sqrt{n}}\right)\sim\|B_nf-f\|_{C[0,1]}+\|B_{kn}f-f\|_{C[0,1]} \tag{1.4}$$

holds, where $\omega^2_\varphi(f,t)=\sup_{0<h\leqslant t}\|\Delta^2_{h\varphi}f\|$, $\varphi(x)=\sqrt{x(1-x)}$.

In [6], Totik extended the Ditzian–Ivanov result to a large family of operators. Typical examples are the Bernstein, Szasz–Mirakjan, Baskakov operators and related ones. In [5] we gave a strong converse inequality on simultaneous approximation for Baskakov–Durrmeyer operators with $\omega^2_\varphi(f^{(2r)},t)$. If we want to deal with $\omega^2_{\varphi^\lambda}(f,t)$, $0\leqslant\lambda\leqslant 1$, it should be noted that the above results are only for $\lambda=1$.

In this paper we deal with $\omega^2_{\varphi^\lambda}(f,t)$ $(0\leqslant\lambda\leqslant 1)$. We obtain a result that is similar to (1.3) (Stechkin–Marchaud inequality) for the Baskakov operator. Though we also attempted to get a result(strong converse inequality) of type (1.4), it was not successful.

For the Baskakov polynomials defined for $f\in C[0,\infty)$ by

$$V_n(f,x)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)v_{n,k}(x),\qquad v_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}. \tag{1.5}$$

By using the method similar to [2], it is not difficult to show

$$|V_n(f,x)-f(x)|\leqslant M\omega^2_{\varphi^\lambda}\left(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right),\qquad 0\leqslant\lambda\leqslant 1,\ \varphi(x)=\sqrt{x(1+x)}. \tag{1.6}$$

The purpose of this paper is to prove the following Stechkin–Marchaud-type inequalities for Baskakov polynomials,

$$\omega^2_{\varphi^\lambda}\left(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)_{\alpha,\beta}\leqslant Mn^{-1}\left(\sum_{k=1}^{n}\|V_kf-f\|_0^*+\|f\|_0^*\right), \tag{1.7}$$

where $\omega^2_{\varphi^\lambda}(f,\varphi^{1-\lambda}(x)/\sqrt{n})_{\alpha,\beta}$, $\|\cdot\|_0^*$ will be defined in next section. It is easy to see that our result is more extensive. It unifies the result of $\omega^2(f,t)$ and $\omega^2_\varphi(f,t)$. As a corollary of the main result, we will give the inverse theorem of (1.6).

2. LEMMAS

Since we only consider the Baskakov operator from now on, let us suppose that $\varphi(x)^2 = x(1+x)$. First, we give some notations,

$$C_0 := \{f \in C[0, \infty), f(0) = 0\},$$
$$C^2 := \{f \in C_0, f'' \in C[0, \infty)\},$$

where $C[0, \infty)$ denotes the set of bounded continuous functions.

For $0 \leqslant \gamma \leqslant 2$,

$$\|f\|_\gamma := \sup_{x \in [0, \infty)} \{|\varphi^{-\gamma}(x) f(x)|\} = \|\varphi^{-\gamma} f\|,$$
$$C_\gamma := \{f \in C_0, \|f\|_\gamma < \infty\},$$
$$C_\gamma^2 := \{f \in C^2, \|f''\|_\gamma < \infty\}.$$

For $0 \leqslant \lambda \leqslant 1$, $0 < \alpha < 2$, $0 \leqslant \beta \leqslant 2$ and $(1-\lambda)\alpha + \beta \leqslant 2$,

$$C^0_{\lambda, \alpha, \beta} := C_{(1-\lambda)\alpha+\beta}, \qquad C^2_{\lambda, \alpha, \beta} := C^2_{(1-\lambda)\alpha+\beta},$$
$$\|f\|_0^* := \|f\|_{(1-\lambda)\alpha+\beta}, \qquad \|f\|_2^* := \|\varphi^2 f''\|_{(1-\lambda)\alpha+\beta}.$$

Here, the notations $\|f\|_0^*$ and $\|f\|_2^*$ are related to α, β and λ. For the sake of brevity we suppress in part the parameters α, β and λ. Our modulus of smoothness is given by

$$\omega^2_{\varphi^\lambda}(f, t)_{\alpha, \beta} := \sup_{0 < h \leqslant t} \{|\varphi^{(\lambda-1)\alpha-\beta}(x)\, \Delta^2_{h\varphi^\lambda} f(x)|, x \pm h\varphi^\lambda(x) \geqslant 0\},$$
$$\Delta_h^2 f(x) := f(x+h) - 2f(x) + f(x-h),$$

and our K-functional by

$$K_\lambda^{\alpha, \beta}(f, t) := \inf_{g \in C^2_{\lambda, \alpha, \beta}} \{\|f-g\|_0^* + t^2 \|g\|_2^*\}.$$

Now, we give some lemmas.

LEMMA 2.1. *For $f \in C_\gamma$, $0 \leqslant \gamma \leqslant 2$, one has*

$$\|\varphi^2 V_n'' f\|_\gamma \leqslant M_0 n \|f\|_\gamma, \tag{2.1}$$
$$\|\varphi^2 V_n'' f\|_2 \leqslant M_0 n^{2-\gamma/2} \|f\|_\gamma. \tag{2.2}$$

Moreover, if $f \in C^2$, then

$$\|\varphi^2 V_n'' f\|_\gamma \leqslant \frac{n+1}{n} \|\varphi^2 f''\|_\gamma + 24 n^{\gamma/2-1} \|\varphi^2 f''\|_2, \tag{2.3}$$

$$\|\varphi^2 V_n'' f\|_2 \leqslant \frac{n+1}{n} \|\varphi^2 f''\|_2. \tag{2.4}$$

Proof. To prove (2.1) we set $E_n = [\frac{A}{n}, \infty)$, where $A > 0$ is a fixed number.

(i) If $x \in E_n^c$, without loss of generality, we may assume $\varphi^2(x) < \frac{1}{n}$. Using the representation of $V_n''(f, x)$ (cf. [4, p. 125]), we write

$$\begin{aligned}
&|\varphi^{2-\gamma}(x)\, V_n''(f, x)| \\
&\quad = \left|\varphi^{2-\gamma}(x)\, n(n+1) \sum_{k=0}^{\infty} v_{n+2,k}(x) \Delta_{1/n}^2 f\left(\frac{k}{n}\right)\right| \\
&\quad \leqslant 2n^{1+\gamma/2} \left|\sum_{k=0}^{\infty} v_{n+2,k}(x) \left(f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right)\right| \\
&\quad \leqslant 2n^{1+\gamma/2} \|f\|_\gamma \left(\sum_{k=0}^{\infty} v_{n+2,k}(x)\, \varphi^\gamma\left(\frac{k+2}{n}\right) + 2 \sum_{k=0}^{\infty} v_{n+2,k}(x)\, \varphi^\gamma\left(\frac{k+1}{n}\right)\right. \\
&\qquad \left. + \sum_{k=0}^{\infty} v_{n+2,k}(x)\, \varphi^\gamma\left(\frac{k}{n}\right)\right),
\end{aligned}$$

where $\Delta_h^2 f(t) = f(t+2h) - 2f(t+h) + f(t)$. We only estimate the first term. Estimates of the other terms are similar. By the Hölder inequality, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} v_{n+2,k}(x)\, \varphi^\gamma\left(\frac{k+2}{n}\right) &\leqslant \left(\sum_{k=0}^{\infty} v_{n+2,k}(x)\, \varphi^2\left(\frac{k+2}{n}\right)\right)^{\gamma/2} \cdot \left(\sum_{k=0}^{\infty} v_{n+2,k}(x)\right)^{1-\gamma/2} \\
&= \left(\sum_{k=1}^{\infty} v_{n+2,k}(x)\, \varphi^2\left(\frac{k+2}{n}\right) + v_{n+2,0}(x)\, \varphi^2\left(\frac{2}{n}\right)\right)^{\gamma/2} \\
&\leqslant M_1 \left(\varphi^2(x) + \frac{1}{n}\right)^{\gamma/2} \\
&\leqslant M_2 n^{-\gamma/2}.
\end{aligned}$$

This leads to (2.1).

(ii) If $x \in E_n$, using (cf [4, p. 127])

$$V_n''(f) = (x(1+x))^{-2} \sum_{i=0}^{2} Q_i^V(x, n)\, n^i \sum_{k=0}^{\infty} v_{n,k}(x) \left|\frac{k}{n} - x\right|^i f\left(\frac{k}{n}\right)$$

and

$$|(x(1+x))^{-2} Q_i^V(x,n)\, n^i| \leqslant C\left(\frac{n}{x(1+x)}\right)^{1+i/2},$$

we have

$$\begin{aligned}|\varphi^{2-\gamma}(x)\, V_n''(f,x)| &\leqslant \sum_{i=0}^{2} \left|\varphi^{-\gamma}(x)\, n\left(\frac{n^{1/2}}{\varphi(x)}\right)^i \sum_{k=0}^{\infty} v_{n,k}(x)\left|\frac{k}{n}-x\right|^i f\left(\frac{k}{n}\right)\right| \\ &\leqslant \sum_{i=0}^{2} n\,\|f\|_\gamma \left|\varphi^{-\gamma}(x)\left(\frac{n^{1/2}}{\varphi(x)}\right)^i \sum_{k=0}^{\infty} v_{n,k}(x)\left|\frac{k}{n}-x\right|^i \varphi^\gamma\left(\frac{k}{n}\right)\right|. \end{aligned} \tag{2.5}$$

By the Hölder inequality,

$$\begin{aligned}&\sum_{k=0}^{\infty} v_{n,k}(x)\left|\frac{k}{n}-x\right|^i \varphi^\gamma\left(\frac{k}{n}\right) \\ &\quad\leqslant \left(\sum_{k=0}^{\infty} v_{n,k}(x)\,\varphi^2\left(\frac{k}{n}\right)\right)^{\gamma/2}\left(\sum_{k=0}^{\infty} v_{n,k}(x)\left|\frac{k}{n}-x\right|^{\frac{i}{1-\gamma/2}}\right)^{1-\gamma/2}. \end{aligned} \tag{2.6}$$

Let the integer m satisfy $2m > \frac{i}{1-\gamma/2}$. We use the Hölder inequality and Lemma 9.4.4 of [4] to obtain

$$\begin{aligned}&\left(\sum_{k=0}^{\infty} v_{n,k}(x)\left|\frac{k}{n}-x\right|^{\frac{i}{1-\gamma/2}}\right)^{1-\gamma/2} \\ &\quad\leqslant \left(\sum_{k=0}^{\infty} v_{n,k}(x)\left|\frac{k}{n}-x\right|^{2m}\right)^{\frac{i}{2m(1-\gamma/2)}(1-\gamma/2)} \left(\sum_{k=0}^{\infty} v_{n,k}(x)\right)^{\left(1-\frac{i}{2m(1-\gamma/2)}\right)(1-\gamma/2)} \\ &\quad= \left(\sum_{k=0}^{\infty} v_{n,k}(x)\left|\frac{k}{n}-x\right|^{2m}\right)^{\frac{i}{2m}} \\ &\quad\leqslant C_1\left(\frac{\varphi(x)}{n^{1/2}}\right)^i. \end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned}\left(\sum_{k=0}^{\infty} v_{n,k}(x)\,\varphi^2\left(\frac{k}{n}\right)\right)^{\gamma/2} &= \left(\frac{n+1}{n}\,\varphi^2(x)\sum_{k=0}^{\infty} v_{n+2,k}(x)\right)^{\gamma/2} \\ &= \left(\frac{n+1}{n}\right)^{\gamma/2}\varphi^\gamma(x) \leqslant 2\varphi^\gamma(x). \end{aligned} \tag{2.8}$$

Combining (2.5), (2.6), (2.7) and (2.8), we obtain (2.1) .

The proof of (2.2) is similar to that of (2.1). Next we prove (2.3). We have

$$\begin{aligned}\varphi^2(x)\,V_n''(f,x)&=n(n+1)\,\varphi^2(x)\sum_{k=0}^{\infty} v_{n+2,k}(x)\,\Delta^2_{1/n}f\left(\frac{k+1}{n}\right)\\&=n^2\sum_{k=1}^{\infty}\varphi^2\left(\frac{k}{n}\right)v_{n,k}(x)\,\Delta^2_{1/n}f\left(\frac{k}{n}\right).\end{aligned}\tag{2.9}$$

Let $y\geqslant 1/n$ and $|u|\leqslant 1/n$. Then

$$\frac{n+1}{n}\varphi^2(y+u)+\frac{12}{n}-\varphi^2(y)=\frac{y}{n}+\frac{u}{n}+\frac{y^2}{n}+\frac{u^2}{n}+\frac{2yu}{n}+u+2yu+u^2+\frac{12}{n}.$$

If $0\leqslant u\leqslant 1/n$, the representation is obviously nonnegative.

Otherwise, it is equal to $(0\leqslant u\leqslant 1/n)$

$$\begin{aligned}\frac{y}{n}-\frac{u}{n}+\frac{y^2}{n}+\frac{u^2}{n}-\frac{2yu}{n}-u-2yu+u^2+\frac{12}{n}&\geqslant\frac{y^2}{n}+\frac{12}{n}-\left(\frac{1}{n^2}+\frac{2y}{n^2}+\frac{1}{n}+\frac{y}{n}\right)\\&\geqslant 0.\end{aligned}$$

Therefore,

$$\varphi^2(y)\leqslant\frac{n+1}{n}\varphi^2(y+u)+\frac{12}{n}.$$

Since the function $t^{1-\gamma/2}(0\leqslant\gamma\leqslant 2)$ is subadditive,

$$\varphi^{2-\gamma}(y)\leqslant\left(\frac{n+1}{n}\right)^{1-\gamma/2}\varphi^{2-\gamma}(y+u)+12n^{\gamma/2-1}.$$

Therefore, for $f\in C^2$,

$$\begin{aligned}&\varphi^{2-\gamma}(y)\,|\Delta^2_{1/n}f(y)|\\&\quad\leqslant\varphi^{2-\gamma}(y)\int_{-1/2n}^{1/2n}\int_{-1/2n}^{1/2n}|f''(y+s+t)|\,ds\,dt\\&\quad\leqslant\left(\frac{n+1}{n}\right)^{1-\gamma/2}\int_{-1/2n}^{1/2n}\int_{-1/2n}^{1/2n}\varphi^{2-\gamma}(y+s+t)\,|f''(y+s+t)|\,ds\,dt\\&\qquad+12n^{\gamma/2-1}\cdot\int_{-1/2n}^{1/2n}\int_{-1/2n}^{1/2n}|f''(y+s+t)|\,ds\,dt\\&\quad\leqslant n^{-2}\left[\left(\frac{n+1}{n}\right)^{1-\gamma/2}\|\varphi^2f''\|_\gamma+12n^{\gamma/2-1}\,\|\varphi^2f''\|_2\right].\end{aligned}\tag{2.10}$$

On the other hand, by the Hölder inequality and (2.8),

$$\sum_{k=0}^{\infty} \varphi^{\gamma}\left(\frac{k}{n}\right) v_{n,k}(x) \leqslant \left(\sum_{k=0}^{\infty} \varphi^{2}\left(\frac{k}{n}\right) v_{n,k}(x)\right)^{\gamma/2} = \left(\frac{n+1}{n}\right)^{\gamma/2} \varphi^{\gamma}(x), \tag{2.11}$$

thus, in view of (2.9), (2.10), and (2.11), we have

$$\begin{aligned}
\varphi^{2-\gamma}(x)\,|V_n''(f,x)| &\leqslant n^2 \varphi^{-\gamma}(x) \sum_{k=1}^{\infty} \varphi^{2-\gamma}\left(\frac{k}{n}\right) \left|\Delta_{1/n}^2 f\left(\frac{k}{n}\right)\right| \varphi^{\gamma}\left(\frac{k}{n}\right) v_{n,k}(x) \\
&\leqslant \left(\frac{n+1}{n}\right)^{\gamma/2} \left(\left(\frac{n+1}{n}\right)^{1-\gamma/2} \|\varphi^2 f''\|_{\gamma} + 12 n^{\gamma/2-1} \|\varphi^2 f''\|_2\right) \\
&\leqslant \frac{n+1}{n} \|\varphi^2 f''\|_{\gamma} + 24 n^{\gamma/2-1} \|\varphi^2 f''\|_2.
\end{aligned}$$

Finally, we prove (2.4). We write

$$\begin{aligned}
|V_n''(f,x)| &= n(n+1) \sum_{k=0}^{\infty} v_{n+2,k}(x) \left|\Delta_{1/n}^2 f\left(\frac{k}{n}\right)\right| \\
&\leqslant n(n+1) \sum_{k=0}^{\infty} v_{n+2,k}(x) \int_0^{1/n} \int_0^{1/n} \left| f''\left(\frac{k}{n} + s + t\right)\right| ds\, dt \\
&\leqslant \frac{n(n+1)}{n^2} \|f''\| \sum_{k=0}^{\infty} v_{n+2,k}(x) \\
&= \frac{n+1}{n} \|f''\|.
\end{aligned}$$

Therefore,

$$\|\varphi^2 V_n'' f\|_2 \leqslant \frac{n+1}{n} \|\varphi^2 f''\|_2.$$

The proof is complete.

LEMMA 2.2 (cf. [7]). *Suppose that for nonnegative sequences* $\{\sigma_n\}$, $\{\tau_n\}$ *with* $\sigma_1 = 0$, *the inequality* ($p > 0$)

$$\sigma_n \leqslant \left(\frac{k}{n}\right)^p \sigma_k + \tau_k \qquad (1 \leqslant k \leqslant n) \tag{2.12}$$

holds for $n \in N$, then

$$\sigma_n \leqslant M_p n^{-p} \sum_{k=1}^{n} k^{p-1} \tau_k. \tag{2.13}$$

LEMMA 2.3 (cf. [7]). *Suppose that for nonnegative sequences $\{\mu_n\}$, $\{\nu_n\}$, $\{\psi_n\}$ with $\mu_1 = 0$ and $\nu_1 = 0$, the inequalities* ($0 < r < s$, $1 \leqslant k \leqslant n$)

$$\mu_n \leqslant \left(\frac{k}{n}\right)^r \mu_k + \nu_k + \psi_k \tag{2.14}$$

and

$$\nu_n \leqslant \left(\frac{k}{n}\right)^s \nu_k + \psi_k \tag{2.15}$$

hold for $n \in N$. Then

$$\mu_n \leqslant M_{r,s} n^{-r} \sum_{k=1}^{n} k^{r-1} \psi_k. \tag{2.16}$$

By Lemma 2.1, 2.2, and 2.3, we can obtain the following lemma.

LEMMA 2.4. *For $f \in C_\gamma$, $0 \leqslant \gamma \leqslant 2$, we have*

$$\|\varphi^2 V_n'' f\|_\gamma \leqslant M \left(\sum_{k=1}^{n} \|V_k f - f\|_\gamma + \|f\|_\gamma \right). \tag{2.17}$$

Proof. If $0 \leqslant \gamma < 2$, let ($n \in N$, $1 \leqslant m \leqslant n$)

$$\mu_m = m^{-1} \|\varphi^2 (V_m'' - V_1'') f\|_\gamma,$$
$$\nu_m = 24 m^{\gamma/2 - 2} \|\varphi^2 (V_m'' - V_1'') f\|_2$$

and

$$\psi_m = 72 M_0 (\|V_m f - f\|_\gamma + n^{-1} \|f\|_\gamma).$$

By (2.1), (2.2) and (2.3), we have

$$\begin{aligned}
\mu_n &\leqslant n^{-1}\,\|\varphi^2 V''_n f\|_\gamma + n^{-1}\,\|\varphi^2 V''_1 f\|_\gamma \\
&\leqslant n^{-1}\,\|\varphi^2 V''_n V_k f\|_\gamma + n^{-1}\,\|\varphi^2 V''_n (V_k f - f)\|_\gamma + n^{-1} M_0\,\|f\|_\gamma \\
&\leqslant n^{-1}\left(\frac{n+1}{n}\,\|\varphi^2 V''_k f\|_\gamma + 24 n^{\gamma/2-1}\,\|\varphi^2 V''_k f\|_2\right) \\
&\quad + M_0\,\|V_k f - f\|_\gamma + M_0 n^{-1}\,\|f\|_\gamma \\
&\leqslant n^{-1}\,\|\varphi^2 V''_k f\|_\gamma + n^{-2}\,\|\varphi^2 V''_k f\|_\gamma + 24 n^{\gamma/2-2}\,\|\varphi^2 V''_k f\|_2 \\
&\quad + M_0(\|V_k f - f\|_\gamma + n^{-1}\,\|f\|_\gamma) \\
&\leqslant n^{-1}\,\|\varphi^2 (V''_k - V''_1)\, f\|_\gamma + n^{-1}\,\|\varphi^2 V''_1 f\|_\gamma + \frac{M_0 k}{n^2}\,\|f\|_\gamma \\
&\quad + 24 n^{\gamma/2-2}\,\|\varphi^2 (V''_k - V''_1)\, f\|_2 + 24 n^{\gamma/2-2}\,\|\varphi^2 V''_1 f\|_2 \\
&\quad + M_0(\|V_k f - f\|_\gamma + n^{-1}\,\|f\|_\gamma) \\
&\leqslant n^{-1}\,\|\varphi^2 (V''_k - V''_1)\, f\|_\gamma + 24 n^{\gamma/2-2}\,\|\varphi^2 (V''_k - V''_1)\, f\|_2 \\
&\quad + 26 M_0 n^{-1}\,\|f\|_\gamma + M_0(\|V_k f - f\|_\gamma + n^{-1}\,\|f\|_\gamma) \\
&\leqslant \frac{k}{n}\,\mu_k + \left(\frac{k}{n}\right)^{2-\gamma/2} \nu_k + 27 M_0(\|V_k f - f\|_\gamma + n^{-1}\,\|f\|_\gamma) \\
&\leqslant \frac{k}{n}\,\mu_k + \nu_k + \psi_k.
\end{aligned}$$

Hence, (2.14) holds for $r = 1$. On the other hand, by (2.2) and (2.4),

$$\begin{aligned}
\nu_n &\leqslant 24 n^{\gamma/2-2}\,\|\varphi^2 V''_n f\|_2 + 24 n^{\gamma/2-2}\,\|\varphi^2 V''_1 f\|_2 \\
&\leqslant 24 n^{\gamma/2-2}\,\|\varphi^2 V''_n V_k f\|_2 + 24 n^{\gamma/2-2}\,\|\varphi^2 V''_n (V_k f - f)\|_2 + 24 M_0 n^{-1}\,\|f\|_\gamma \\
&\leqslant 24 n^{\gamma/2-2}\,\frac{n+1}{n}\,\|\varphi^2 V''_k f\|_2 + 24 M_0\,\|V_k f - f\|_\gamma + 24 M_0 n^{-1}\,\|f\|_\gamma \\
&\leqslant 24 n^{\gamma/2-2}\,\|\varphi^2 V''_k f\|_2 + 24 n^{\gamma/2-2} n^{-1}\,\|\varphi^2 V''_k f\|_2 + 24 M_0(\|V_k f - f\|_\gamma + n^{-1}\,\|f\|_\gamma) \\
&\leqslant 24 n^{\gamma/2-2}\,\|\varphi^2 (V''_k - V''_1)\, f\|_2 + 24 n^{\gamma/2-2}\,\|\varphi^2 V''_1 f\|_2 \\
&\quad + 24 M_0 \left(\frac{k}{n}\right)^{2-\gamma/2} n^{-1}\,\|f\|_\gamma + 24 M_0(\|V_k f - f\|_\gamma + n^{-1}\,\|f\|_\gamma) \\
&\leqslant 24 n^{\gamma/2-2}\,\|\varphi^2 (V''_k - V''_1)\, f\|_2 + 48 M_0 n^{-1}\,\|f\|_\gamma + 24 M_0(\|V_k f - f\|_\gamma + n^{-1}\,\|f\|_\gamma) \\
&\leqslant \left(\frac{k}{n}\right)^{2-\gamma/2} \nu_k + \psi_k,
\end{aligned}$$

and (2.15) holds for $s = 2-\gamma/2$. Therefore Lemma 2.3 implies

$$\|\varphi^2(V''_n - V''_1) f\|_\gamma \leqslant M_1 \sum_{k=1}^{n} (\|V_k f - f\|_\gamma + n^{-1} \|f\|_\gamma)$$

$$= M_1 \left(\sum_{k=1}^{n} \|V_k f - f\|_\gamma + \|f\|_\gamma \right).$$

Therefore,

$$\|\varphi^2 V''_n f\|_\gamma \leqslant M_1 \left(\sum_{k=1}^{n} \|V_k f - f\|_\gamma + \|f\|_\gamma \right) + M_0 \|f\|_\gamma$$

$$= M \left(\sum_{k=1}^{n} \|V_k f - f\|_\gamma + \|f\|_\gamma \right).$$

Concerning the case $\gamma = 2$, we apply Lemma 2.2 with $p = 1$ to $(1 \leqslant m \leqslant n)$

$$\sigma_m = m^{-1} \|\varphi^2(V''_m - V''_1) f\|_2 \qquad \text{and} \qquad \tau_m = 3M_0(\|V_m f - f\|_2 + n^{-1} \|f\|_2),$$

which implies (2.17) analogously (cf (2.2), (2.4)).

To establish our main theorem, we need the following lemma.

LEMMA 2.5. *If $0 \leqslant \gamma \leqslant 2$, $t > 0$, $x \geqslant t$ and either of*

$$\text{(i)} \quad 0 < t < 1, \qquad \text{(ii)} \quad x \geqslant 2t$$

is satisfied, then we have

$$\int_{-t/2}^{t/2} \int_{-t/2}^{t/2} \varphi^\gamma(x+u+v)\, du\, dv \leqslant M t^2 \varphi^{-\gamma}(x).$$

Proof. If condition (i) is satisfied, then for $\gamma = 2$, it is known (cf. [1]). For $0 \leqslant \gamma < 2$, we use the Hölder inequality

$$\int_{-t/2}^{t/2} \int_{-t/2}^{t/2} \varphi^{-\gamma}(x+u+v)\, du\, dv$$

$$\leqslant \left(\int_{-t/2}^{t/2} \int_{-t/2}^{t/2} \varphi^{-2}(x+u+v)\, du\, dv \right)^{\gamma/2} \left(\int_{-t/2}^{t/2} \int_{-t/2}^{t/2} du\, dv \right)^{1-\gamma/2}$$

$$\leqslant M(t^2 \varphi^{-2}(x))^{\gamma/2}\, t^{2(1-\gamma/2)}$$

$$\leqslant M t^2 \varphi^{-\gamma}(x).$$

If condition (ii) is satisfied, then $x-t \geqslant \frac{1}{2}x$. Therefore,

$$\int_{-t/2}^{t/2}\int_{-t/2}^{t/2} \varphi^{-\gamma}(x+u+v)\,du\,dv \leqslant t^2\varphi^{-\gamma}(x-t) \leqslant t^2\varphi^{-\gamma}\left(\frac{x}{2}\right) \leqslant Mt^2\varphi^{-\gamma}(x).$$

3. MAIN THEOREMS AND COROLLARIES

Now, we prove the main theorems.

THEOREM 3.1. *Suppose $f \in C^0_{\lambda,\alpha,\beta}$. Then one has*

$$K_\lambda^{\alpha,\beta}\left(f, \frac{1}{n}\right) \leqslant Cn^{-1}\left(\sum_{k=1}^{n} \|V_k f - f\|_0^* + \|f\|_0^*\right). \tag{3.1}$$

Proof. For $n \geqslant 2$, there exists $l \in N$, such that $n/2 \leqslant l \leqslant n$, and

$$\|V_l f - f\|_0^* \leqslant \|V_k f - f\|_0^* \left(\frac{n}{2} \leqslant k \leqslant n\right).$$

On the other hand, Lemma 2.4 implies (where we set $\gamma = (1-\lambda)\alpha + \beta$)

$$\|V_n'' f\|_2^* \leqslant M\left(\sum_{k=1}^{n} \|V_k f - f\|_0^* + \|f\|_0^*\right).$$

Therefore, using the definition of $K_\lambda^{\alpha,\beta}(f, \frac{1}{n})$, we have

$$\begin{aligned}
K_\lambda^{\alpha,\beta}\left(f, \frac{1}{n}\right) &\leqslant \|V_l f - f\|_0^* + \frac{1}{n}\|V_l f\|_2^* \\
&\leqslant \frac{2}{n}\sum_{k=n/2}^{n} \|V_k f - f\|_0^* + \frac{1}{n}M\left(\sum_{k=1}^{l} \|V_k f - f\|_0^* + \|f\|_0^*\right) \\
&\leqslant \frac{2}{n}\left(\sum_{k=1}^{n} \|V_k f - f\|_0^* + \|f\|_0^*\right) + \frac{1}{n}M\left(\sum_{k=1}^{n} \|V_k f - f\|_0^* + \|f\|_0^*\right) \\
&\leqslant C\frac{1}{n}\left(\sum_{k=1}^{n} \|V_k f - f\|_0^* + \|f\|_0^*\right).
\end{aligned}$$

The proof is complete.

Remark. Wickeren's Proof in [7] is followed for Theorem 3.1, and from (3.1) we can deduce the following theorem.

THEOREM 3.2. *Suppose $f \in C^0_{\lambda,\alpha,\beta}$. Then one has*

$$\omega^2_{\varphi^\lambda}\left(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)_{\alpha,\beta} \leqslant C\frac{1}{n}\left(\sum_{k=1}^{n} \|V_k f - f\|_0^* + \|f\|_0^*\right). \tag{3.2}$$

Proof. According to the definition of $K_\lambda^{\alpha,\beta}(f, \frac{1}{n})$, there exists $g \in C^2_{\lambda,\alpha,\beta}$ such that

$$\|f-g\|_0^* + \frac{1}{n}\|g\|_2^* \leqslant 2K_\lambda^{\alpha,\beta}\left(f, \frac{1}{n}\right). \tag{3.3}$$

On the other hand,

$$|\Delta^2_{h\varphi^\lambda} f(x)| \leqslant |\Delta^2_{h\varphi^\lambda}(f-g)(x)| + |\Delta^2_{h\varphi^\lambda} g(x)|. \tag{3.4}$$

For the first term of (3.4), since $\varphi^{\alpha(1-\lambda)+\beta}(x)$ is a monotone increasing function, and $x \geqslant h\varphi^\lambda(x)$,

$$\begin{aligned}|\Delta^2_{h\varphi^\lambda}(f-g)(x)| &\leqslant \|f-g\|_0^* \,(\varphi^{\alpha(1-\lambda)+\beta}(x+h\varphi^\lambda(x)) + 2\varphi^{\alpha(1-\lambda)+\beta}(x)\\ &\quad + \varphi^{\alpha(1-\lambda)+\beta}(x-h\varphi^\lambda(x)))\\ &\leqslant \|f-g\|_0^* \,(\varphi^{\alpha(1-\lambda)+\beta}(2x) + 3\varphi^{\alpha(1-\lambda)+\beta}(x))\\ &\leqslant 7\varphi^{\alpha(1-\lambda)+\beta}(x)\,\|f-g\|_0^*.\end{aligned}$$

For the second term of (3.4), we have

$$\begin{aligned}|\Delta^2_{h\varphi^\lambda(x)} g(x)| &= \left|\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2}\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2} g''(x+\mu+v)\,d\mu\,dv\right|\\ &\leqslant \|g\|_2^* \cdot \left|\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2}\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2} \varphi^{-2+(1-\lambda)\alpha+\beta}(x+\mu+v)\,d\mu\,dv\right|.\end{aligned}$$

Set $t = h\varphi^\lambda(x)$. Since $x \geqslant h\varphi^\lambda(x)$, if $x < 1$, one has $0 < t < 1$, which satisfies (i) of Lemma 2.5.

If $x \geqslant 1$, let $h \leqslant \varphi^{1-\lambda}(x)/\sqrt{n}$ $(n \geqslant 8)$. Then

$$t \leqslant \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\varphi^\lambda(x) = \frac{\varphi(x)}{\sqrt{n}} \leqslant \sqrt{\frac{2}{n}}\,x \leqslant \frac{x}{2},$$

which satisfies (ii) of Lemma 2.5. Therefore, suppose $h \leqslant \varphi^{1-\lambda}(x)/\sqrt{n}$ $(n \geqslant 8)$. By Lemma 2.5, we have

$$\begin{aligned}\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2}\int_{-h\varphi^\lambda(x)/2}^{h\varphi^\lambda(x)/2} \varphi^{-2+(1-\lambda)\alpha+\beta}(x+\mu+v)\,d\mu\,dv &\leqslant M(h\varphi^\lambda(x))^2\,\varphi^{-2+(1-\lambda)\alpha+\beta}(x)\\ &\leqslant M\frac{1}{n}\varphi^{(1-\lambda)\alpha+\beta}(x).\end{aligned}$$

Thus, if $h \leqslant \varphi^{1-\lambda}(x)/\sqrt{n}$ $(n \geqslant 8)$, we have

$$|\Delta^2_{\varphi^\lambda} f(x)| \leqslant M_1 \varphi^{\alpha(1-\lambda)+\beta}(x) \left(\|f-g\|^*_o + \frac{1}{n} \|g\|^*_2 \right).$$

Therefore,

$$\omega^2_{\varphi^\lambda}\left(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)_{\alpha,\beta} \leqslant 2M_1 K^{\alpha,\beta}_{\lambda}\left(f, \frac{1}{n}\right) \leqslant C\frac{1}{n}\left(\sum_{k=1}^{n} \|V_k f - f\|^*_0 + \|f\|^*_0\right).$$

However, if $n \leqslant 8$, the result is obvious. This completes the proof of this theorem.

Now, we give some corollaries.

Corollary 3.1. *Let $\lambda = 1$, $0 \leqslant \beta \leqslant 2$, then for $f \in C_\beta$*

$$\omega^2_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_\beta \leqslant C\frac{1}{n}\left(\sum_{k=1}^{n} \|\varphi^{-\beta}(V_k f - f)\| + \|\varphi^{-\beta} f\|\right).$$

This result corresponds to the result of [7] with $\beta = 0$ which is the result of Theorem 9.3.6 in [4] for $s = 1$.

Corollary 3.2. *Let $\lambda = 0$, $0 \leqslant \alpha + \beta = \gamma \leqslant 2$. Then for $f \in C_\gamma$*

$$\omega^2\left(f, \frac{\varphi(x)}{\sqrt{n}}\right)_\gamma \leqslant C\frac{1}{n}\left(\sum_{k=1}^{n} \|\varphi^{-\gamma}(V_k f - f)\| + \|\varphi^{-\gamma} f\|\right).$$

This is a result for the classical modulus.

Corollary 3.3. *For $0 < \alpha < 2$, $0 \leqslant \beta \leqslant 2$, we have the following inverse results*

$$|(V_n f - f)(x)| = O\left(\left(\frac{\varphi(x)}{\sqrt{n}}\right)^\alpha\right) \Rightarrow \omega^2(f, t) = O(t^\alpha), \tag{3.5}$$

$$|(V_n f - f)(x)| = O\left(\left(\frac{1}{\sqrt{n}}\right)^\alpha\right) \Rightarrow \omega^2_\varphi(f, t) = O(t^\alpha), \tag{3.6}$$

$$\varphi^{-\beta}(x)\,|(V_n f - f)(x)| \leqslant M n^{-\alpha/2}$$
$$\Rightarrow \varphi^{-\beta}(x)\,|f(x+t) - 2f(x) + f(x-t)| \leqslant M \frac{t^\alpha}{\varphi^\alpha(x)}, \tag{3.7}$$

$$|(V_n f - f)(x)| = O\left(\left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)^\alpha\right) \Rightarrow \omega^2_{\varphi^\lambda}(f, t) = O(t^\alpha). \tag{3.8}$$

Proof. Since the proof of (3.5), (3.6), and (3.7) are similar, here we only prove (3.7).

Applying Corollary 3.2, let $\gamma = \beta$, $\varphi(x)/\sqrt{n+1} < t \leqslant \varphi(x)/\sqrt{n}$. Then

$$\begin{aligned}
&\varphi^{-\beta}(x)\,|f(x+t) - 2f(x) + f(x-t)| \\
&\quad \leqslant \omega^2(f, t)_\beta \leqslant \omega^2\left(f, \frac{\varphi(x)}{\sqrt{n}}\right)_\beta \\
&\quad \leqslant C\left(n^{-1} \sum_{k=1}^{n} \|\varphi^{-\beta}(V_k f - f)\| + (n^{-1}\,\|\varphi^{-\beta} f\|\right) \\
&\quad \leqslant C_1 n^{-1} \sum_{k=1}^{n} k^{-\alpha/2} + C_2 n^{-1} \\
&\quad \leqslant C_3 n^{-\alpha/2} \leqslant M \frac{t^\alpha}{\varphi^\alpha(x)}.
\end{aligned}$$

Last, we prove (3.8). In Theorem 3.2, let $\beta = 0$; thus

$$\begin{aligned}
\|V_n f - f\|_0^* &= \sup_x \,\{\varphi^{(\lambda-1)\alpha}(x)\,|(V_n f - f)(x)|\} \\
&\leqslant M n^{-\alpha/2}.
\end{aligned}$$

Let $\varphi^{1-\lambda}(x)/\sqrt{n+1} < t \leqslant \varphi^{1-\lambda}(x)/\sqrt{n}$. Then $(h \leqslant t)$

$$\begin{aligned}
&\varphi^{\alpha(\lambda-1)}(x)\,|f(x + h\varphi^\lambda(x)) - 2f(x) + f(x - h\varphi^\lambda(x))| \\
&\quad \leqslant C n^{-1}\left(\sum_{k=1}^{n} M k^{-\alpha/2} + \|f\|_0^*\right) \\
&\quad \leqslant M_1 n^{-\alpha/2}.
\end{aligned}$$

Therefore,

$$|\Delta^2_{h\varphi^\lambda} f(x)| \leqslant M_1 \left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)^\alpha \leqslant M_2 t^\alpha.$$

This is

$$\omega^2_{\varphi^\lambda}(f, t) \leqslant M_2 t^\alpha.$$

The proof is complete.

Remark. (1) Relation (3.8) is the inverse theorem of (1.6).

(2) Since $V_n(f, x)$ preserves constants, the condition $f(0) = 0$ can be omitted in the results.

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